An Open Problem in Dynamic Edge Coloring

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1 The Setting

Let $G = (V, E)$ be a simple, undirected graph with n vertices, m edges and maximum degree Δ. It was shown by [Viz64] that the graph G can always be *edge colored* with at most $\Delta + 1$ colors. Furthermore, such a coloring can be computed in near-linear time in the *static* setting [ABB+24].

The dynamic setting: In the *dynamic* setting, we have a fixed set of n vertices V, and the graph G evolves over time via a sequence of edge insertions and deletions. We assume that we have some fixed value Δ and that the maximum degree of G is at most Δ at all times. Our objective is to design an algorithm that *explicitly* maintains an edge coloring χ of G at all times and has a small *update time*, which is the time it takes to update the edge coloring χ after an edge is either inserted or deleted from the graph G. We also define the *recourse* of an update as the number of edges that change color while handling the update. Since our algorithm explicitly maintains the edge coloring χ, the recourse of an update is a lower bound on the time it takes to handle that update.

The state-of-the-art for dynamic edge coloring: It is known how to maintain a $(1+\epsilon)\Delta$-edge coloring in $\tilde{O}(\text{poly}(1/\epsilon))$ worst-case update time against an adaptive adversary [DHZ19, Chr23] and in $O(\text{poly}(1/\epsilon))$ amortized update time against an oblivious adversary [BCPS24].

2 Open Problems

It is natural to ask if one can improve on these results. As a first step, we can ignore update time and consider only recourse, and see whether or not we can maintain a $(\Delta + o(\Delta))$-edge coloring with only $\tilde{O}(1)$ recourse. More specifically, one can ask the following question.

Q1: Is it possible to maintain a $(\Delta + \tilde{O}(\Delta^{1-\gamma}))$-edge coloring of a dynamic graph G with $\tilde{O}(1)$ recourse, for $\gamma > 0$?

This question is open even considering *amortized recourse* and an *oblivious adversary*. By a standard reduction, we can also assume that the graph is bipartite [KS87].

3 Possible Hardness

A result of [CHL+20] shows that there exists a graph G such that, for any $\mu = O(\Delta)$, there exists a $(\Delta + \mu)$-edge coloring χ of G with an uncolored edge e such that extending χ to e requires changing the colors of $\tilde{\Omega}(\Delta/\mu)$ edges. This implies that any analysis which does not consider the structure of the edge coloring maintained by a dynamic algorithm will not be sufficient to answer Q1. We remark that many of the current state-of-the-art dynamic algorithms for this problem are analysed in this way, and assume that the edge coloring of the graph is completely arbitrary at the start of each update [DHZ19, Chr23]. If this lower bound can be extended to get an analogous bound in the dynamic setting, it would essentially rule out the possibility of obtaining faster dynamic algorithms for this problem.

References

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